Diploma Programme
Programme du diplôme
Programa del Diploma

Markscheme

May 2015

Discrete mathematics

Higher level

Paper 3

12 pages

International Baccalaureate
Baccalauréat International
Bachillerato Internacional

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a valid **Method**; working must be seen.

(M) Marks awarded for **Method**; may be implied by **correct** subsequent working.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

(A) Marks awarded for an **Answer** or for **Accuracy**; may be implied by **correct** subsequent working.

R Marks awarded for clear **Reasoning**.

N Marks awarded for **correct** answers if **no** working shown.

AG Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Mark according to RM™ Assessor instructions and the document "**Mathematics HL: Guidance for e-marking May 2015**". It is essential that you read this document before you start marking. In particular, please note the following:

- Marks must be recorded using the annotation stamps. Please check that you are entering marks for the right question.
- If a part is **completely correct**, (and gains all the "must be seen" marks), use the ticks with numbers to stamp full marks.
- If a part is completely wrong, stamp ***A0*** by the final answer.
- If a part gains anything else, it **must** be recorded using **all** the annotations.
- All the marks will be added and recorded by RM™ Assessor.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *eg* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*eg* substitution into a formula) and ***A1*** for using the **correct** values.
- Where the markscheme specifies ***(M2)***, ***N3***, *etc.*, do **not** split the marks.
- Once a correct answer to a question or part-question is seen, ignore further correct working. However, if further working indicates a lack of mathematical understanding do not award the final ***A1***. An exception to this may be in numerical answers, where a correct exact value is followed by an incorrect decimal. However, if the incorrect decimal is carried through to a subsequent part, and correct ***FT*** working shown, award ***FT*** marks as appropriate but do not award the final ***A1*** in that part.

Examples

	Correct answer seen	Further working seen	Action
1.	$8\sqrt{2}$	5.65685... *(incorrect decimal value)*	Award the final ***A1*** *(ignore the further working)*
2.	$\frac{1}{4}\sin 4x$	$\sin x$	Do not award the final ***A1***
3.	$\log a - \log b$	$\log(a-b)$	Do not award the final ***A1***

3 ***N* marks**

*Award **N** marks for **correct** answers where there is **no** working.*

- Do **not** award a mixture of ***N*** and other marks.
- There may be fewer ***N*** marks available than the total of ***M***, ***A*** and ***R*** marks; this is deliberate as it penalizes candidates for not following the instruction to show their working.

4 **Implied marks**

*Implied marks appear in **brackets eg (M1)**, and can only be awarded if **correct** work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

5 **Follow through marks**

*Follow through (**FT**) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s). To award **FT** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part.*

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks.
- If the error leads to an inappropriate value (*eg* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).
- Within a question part, once an error is made, no further **dependent *A*** marks can be awarded, but ***M*** marks may be awarded if appropriate.
- Exceptions to this rule will be explicitly noted on the markscheme.

6 **Mis-read**

*If a candidate incorrectly copies information from the question, this is a mis-read (**MR**).*
*A candidate should be penalized only once for a particular mis-read. Use the **MR** stamp to indicate that this has been a misread. Then deduct the first of the marks to be awarded, even if this is an **M** mark, but award all others so that the candidate only loses one mark.*

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*eg* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).

7 Discretionary marks *(d)*

*An examiner uses discretion to award a mark on the rare occasions when the markscheme does not cover the work seen. In such cases the annotation DM should be used and a brief **note** written next to the mark explaining this decision.*

8 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If in doubt, contact your team leader for advice.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for part-questions are indicated by **EITHER . . . OR**.
- Where possible, alignment will also be used to assist examiners in identifying where these alternatives start and finish.

9 Alternative forms

*Unless the question specifies otherwise, **accept** equivalent forms.*

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates often do not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

Example: for differentiating $f(x) = 2\sin(5x-3)$, the markscheme gives:

$$f'(x) = \left(2\cos(5x-3)\right)5 \quad \left(=10\cos(5x-3)\right)$$ ***A1***

Award ***A1*** for $\left(2\cos(5x-3)\right)5$, even if $10\cos(5x-3)$ is not seen.

10 Accuracy of Answers

*Candidates should **NO LONGER** be penalized for an accuracy error **(AP)**.*

*If the level of accuracy is specified in the question, a mark will be allocated for giving the answer to the required accuracy. When this is not specified in the question, all numerical answers should be given exactly or correct to three significant figures. Please check work carefully for **FT**.*

11 Crossed out work

If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work.

12 Calculators

A GDC is required for paper 3, but calculators with symbolic manipulation features (for example, TI-89) are not allowed.

Calculator notation
The Mathematics HL guide says:

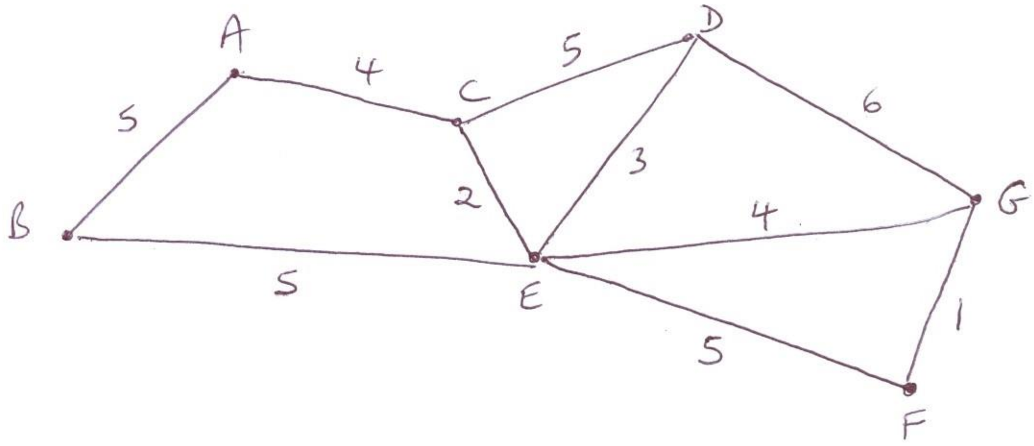
Students must always use correct mathematical notation, not calculator notation.

Do **not** accept final answers written using calculator notation. However, do not penalize the use of calculator notation in the working.

13 More than one solution

Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise.

1. (a) (i)



A2

> **Note:** Award ***A1*** if one edge is missing. Award ***A1*** if the edge weights are not labelled.

(ii) the edges are added in the order:

FG 1 ***M1A1***
CE 2 ***A1***
ED 3
EG 4 ***A1***
AC 4

> **Note:** EG and AC can be added in either order.

(Reject EF)
(Reject CD)
EB 5 **OR** AB 5 ***A1***

> **Notes:** The minimum spanning tree does not have to be seen.
> If only a tree is seen, the order by which edges are added must be clearly indicated.

(iii) 19 ***A1***

[8 marks]

(b) (i) *eg*, PQRSRTSTQP **OR** PQTSTRSRQP ***M1A1***

> **Note:** Award ***M1*** if in either (i) or (ii), it is recognised that edge PQ is needed twice.

(ii) total weight $= 34$ ***A1***

[3 marks]

(c) (i) to determine a cycle where each vertex is visited once only (Hamiltonian cycle) ***A1***
of least total weight ***A1***

continued…

Question 1 continued

(ii) **EITHER**

to reach P, Q must be visited twice which contradicts the definition of the TSP ***R1***

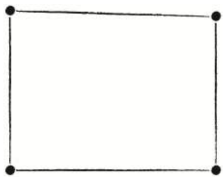
OR

the graph is not a complete graph and hence there is no solution to the TSP ***R1***

[3 marks]

Total [14 marks]

2. (a)



or equivalent ***A1A1***

Note: Award ***A1*** for a correct version of $K_{2,2}$ and ***A1*** for a correct planar representation.

[2 marks]

(b)

A1

[1 mark]

(c)

A1

[1 mark]

(d) the complete bipartite graph $K_{m,n}$ has two subsets of vertices A and B, such that each element of A is connected to every element of B ***A1***
in the complement, no element of A is connected to any element of B. The complement is not a connected graph ***A1***
by definition a tree is connected ***R1***
hence the complement of any complete bipartite graph does not possess a spanning tree ***AG***

[3 marks]

Total [7 marks]

3. (a) **METHOD 1**

attempting to find a solution in the form $u_n = A7^n + B$ ***M1***

EITHER

eg, $u_0 = 5 \Rightarrow 5 = A + B$ and $u_1 = 29 \Rightarrow 29 = 7A + B$ ***A1***

OR

$A7^{n+1} + B = A7^{n+1} + 7B - 6$ (or equivalent) ***A1***

THEN

attempting to solve for A and B ***(M1)***
$u_n = 4 \times 7^n + 1$ ***A1A1***

Note: Accept $A = 4$, $B = 1$ provided the first ***M1*** is awarded.

METHOD 2

attempting an iterative method *eg*, $u_1 = 7(5) - 6$ and
$u_2 = 7^2(5) - 6(7+1)$ (*etc*) ***(M1)***

$u_n = 5 \times 7^n - 6\left(\frac{7^n - 1}{7 - 1}\right)$ ***M1A1***

Note: Award ***M1*** for attempting to express u_n in terms of n.

$u_n = 4 \times 7^n + 1$ ***A1A1***

METHOD 3

attempting to find a solution in the form $u_n = A7^n + B$ ***M1***
$A(n+1) + B = 7(An + B) - 6$
$7B - 6 = B$ ***A1***
attempting to solve for A ***(M1)***
$u_n = 4 \times 7^n + 1$ ***A1A1***

continued…

Question 3 continued

METHOD 4

$u_{n+1} - 7u_n + 6 - (u_n - 7u_{n-1} + 6) = 0 \Rightarrow u_{n+1} - 8u_n + 7u_{n-1} = 0$
$r^2 - 8r + 7 = 0$
$r = 1, 7$

attempting to find a solution in the form $u_n = A7^n + B$ ***M1***

EITHER

eg, $u_0 = 5 \Rightarrow 5 = A + B$ and $u_1 = 29 \Rightarrow 29 = 7A + B$ ***A1***

OR

$A7^{n+1} + B = A7^{n+1} + 7B - 6$ (or equivalent) ***A1***

THEN

attempting to solve for A and B ***(M1)***
$u_n = 4 \times 7^n + 1$ ***A1A1***

[5 marks]

(b) attempting to find the auxiliary equation ***M1***
$r^2 - 10r - 11 = 0$ $\left((r-11)(r+1) = 0\right)$ ***A1***
$r = 11,\ r = -1$ ***A1***
$v_n = A11^n + B(-1)^n$ ***(M1)***
attempting to use the initial conditions ***M1***
$A + B = 4,\ 11A - B = 44$ ***A1***
$v_n = 4 \times 11^n$ ***A1***

[7 marks]

(c) $v_n - u_n = 4\left(11^n - 7^n\right) - 1$ ***M1***

EITHER

$= 4(11-7)\left(11^{n-1} + \ldots + 7^{n-1}\right) - 1$ ***M1A1***

OR

$= 4\left((7+4)^n - 7^n\right) - 1$ ***A1***

subtracting the 7^n from the expanded first bracket ***M1***

THEN

obtaining 16 times a whole number -1 ***A1***
$v_n - u_n \equiv 15 (\text{mod} 16),\ n \in \mathbb{N}$ ***AG***

[4 marks]

Total [16 marks]

4. (a) (i) **METHOD 1**

attempting to use $f = e - v + 2$ and $e \le 3v - 6$ (if $v > 2$) ***(M1)***

$2e \le 6v - 12 = 6(e - f + 2) - 12$ ***M1A1***

leading to $2e \ge 3f$ ***AG***

METHOD 2

each face is bounded by at least three edges ***A1***

Note: Award ***A1*** for stating $e \ge 3f$.

each edge either separates two faces or, if an edge is interior to a face, it gets counted twice ***R1***

Note: Award ***R1*** for stating that each edge contributes two to the sum of the degrees of the faces (or equivalent) *ie*, $\sum \deg(F) = 2e$.

adding up the edges around each face ***R1***

leading to $2e \ge 3f$ ***AG***

(ii) K_5 has $e = 10$ ***A1***

if the graph is planar, $f = 7$ ***A1***

this contradicts the inequality obtained above ***R1***

hence the graph is non-planar ***AG***

[6 marks]

(b) (i) the sum of the vertex degrees $= 2e$ (or is even) or equivalent ***A1***

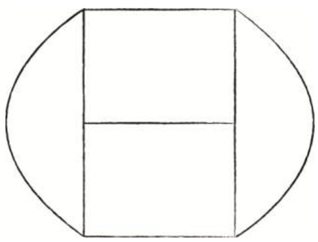
(ii) if each vertex has degree 2, then $2v = 2e$ ***A1***

substituting $v = e$ into Euler's formula ***M1***

$f = 2$ ***A1***

[4 marks]

(c) for example,



A2

[2 marks]

Total [12 marks]

5. (a) every positive integer, greater than 1, is either prime or can be expressed uniquely as a product of primes ***A1A1***

Note: Award ***A1*** for "product of primes" and ***A1*** for "uniquely".

[2 marks]

(b) $5577 = 3 \times 11 \times 13^2$ and $99099 = 3^2 \times 7 \times 11^2 \times 13$ ***M1***

$\gcd(5577, 99099) = 3 \times 11 \times 13$, $\text{lcm}(5577, 99099) = 3^2 \times 7 \times 11^2 \times 13^2$ ***A1A1***

[3 marks]

(c) **METHOD 1**

$n = p_1^{k_1} p_2^{k_2} \ldots p_r^{k_r}$ and $m = p_1^{j_1} p_2^{j_2} \ldots p_r^{j_r}$ ***M1***

employing all the prime factors of n and m, and inserting zero exponents if necessary ***R1***

define $g_i = \min(k_i, j_i)$ and $h_i = \max(k_i, j_i)$ for $i = 1 \ldots r$ ***(M1)***

$\gcd(n, m) = p_1^{g_1} p_2^{g_2} \ldots p_r^{g_r}$ and $\text{lcm}(n, m) = p_1^{h_1} p_2^{h_2} \ldots p_r^{h_r}$ ***A1A1***

noting that $g_i + h_i = k_i + j_i$ for $i = 1 \ldots r$ ***R1***

$\gcd(n, m) \times \text{lcm}(n, m) = n \times m$ for all $n, m \in \mathbb{Z}^+$ ***AG***

METHOD 2

let m and n be expressed as a product of primes where $m = ab$ and $n = ac$ ***M1***

a denotes the factors that are common and b, c are the disjoint factors that are not common ***R1***

$\gcd(n, m) = a$ ***A1***

$\text{lcm}(n, m) = \gcd(n, m)bc$ ***A1***

$\gcd(n, m) \times \text{lcm}(n, m) = a \times (abc)$ ***M1***

$= ab \times ac$ and $m = ab$ and $n = ac$ so ***R1***

$\gcd(n, m) \times \text{lcm}(n, m) = n \times m$ for all $n, m \in \mathbb{Z}^+$ ***AG***

[6 marks]

Total [11 marks]